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Creation and annihilation operators, symmetry and supersymmetry of the 3D isotropic harmonic oscillator

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Abstract
We show that the supersymmetric radial ladder operators of the three-dimensional isotropic harmonic oscillator are contained in the spherical components of the creation and annihilation operators of the system. Also, we show that the constants of motion of the problem, written in terms of these spherical components, lead us to second-order radial operators. Further, we show that these operators change the orbital angular momentum quantum number by two units and are equal to those obtained by the Infeld–Hull factorization method.

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1. Introduction

It is well known that the harmonic oscillator and the hydrogen atom have played an important role in classical and quantum mechanics. Moreover, these problems can be studied following different approaches. Among them are the factorization methods, those of Schrödinger [1] and Infeld–Hull (IH) [2] being the oldest and more common. Supersymmetric quantum mechanics is the most recent way to study solvable as well as perturbative problems, as is extensively shown in [3, 4]. For some quantum problems the relation between constants of motion, ladder operators and supersymmetry has been studied by several authors [5–14]. For example, the SUSY operators for the two- and three-dimensional hydrogen atom are equal to those of the IH factorization method and are contained in their conservative quantities [7, 11, 13]. The simplicity of the two-dimensional hydrogen atom inductively allows us

[4] On leave of absence from[3].

to deduce the well-known Laplace–Runge–Lenz vector (LRLV) from the supersymmetric approach [11, 13]. But because of the complexity of the three-dimensional case, this procedure is no longer appropriate. However, the inverse procedure can be applied, i.e. by going from LRLV to the radial SUSY operators, as has been shown in [7, 11]. In this work, we show for the three-dimensional isotropic harmonic oscillator (3-DIHO) that the SUSY operators coincide with the radial parts of the spherical creation and annihilation operators, and that they are different to those of the IH factorization method. We show that the constants of motion of the problem, written in terms of these spherical components, contain second-order radial operators. Also, we show that these operators change the orbital angular momentum quantum number by two units in the wavefunction and are equal to those obtained by the IH factorization method. The sequence of this work is as follows. In section 2, from the spherical creation and annihilation operators of the 3-DIHO we obtain radial ladder operators that change both quantum numbers, the principal quantum number and orbital angular momentum quantum number. In section 3, we obtain the supersymmetric radial operators and show that they are equal to those we obtained in section 2. In section 4, we show that the constants of motion of the 3-DIHO, identified with $SU(3)$-symmetrized operators, contain the radial operators of the IH factorization method of the 3-DIHO. In section 5, we give concluding remarks.

2. The radial spherical creation and annihilation operators of the 3-DIHO

In what follows we will use $\hbar = \omega = \mu \equiv 1$, where μ is the mass of the particle and ω is the angular frequency of the harmonic oscillator. From the vector creation and annihilation operators,

$$\mathbf{a} = (\mathbf{r} + \mathrm{i}\mathbf{p})/2 \qquad \mathbf{a}^\dagger = (\mathbf{r} - \mathrm{i}\mathbf{p})/2 \tag{1}$$

we can write the Hamiltonian operator of the 3-DIHO as

$$H = \mathbf{a}^\dagger \cdot \mathbf{a} + 3/2 = (\mathbf{p}^2 + \mathbf{r}^2)/2 \tag{2}$$

where $\mathbf{r}$ and $\mathbf{p}$ are the position and the momentum vectors of the particle, respectively, and $\dagger$ denotes Hermitian conjugate. Creation and annihilation operators and the Hamiltonian satisfy the following commutation relations [15]:

$$[a_i, a_j] = \delta_{ij} \qquad \left[H, a_j^\dagger\right] = a_j^\dagger \qquad [H, a_j] = -a_j. \tag{3}$$

The spherical components of $\mathbf{a}$, $a_{\pm 1} \equiv \frac{1}{2}(a_1 \pm \mathrm{i}a_2)$ and $a_0 \equiv a_3$, as well as those of $\mathbf{a}^\dagger$ can be written as

$$\begin{aligned} a_{\pm 1} &= \frac{1}{\sqrt{2}}\left[r\sin\theta\,\mathrm{e}^{\pm\mathrm{i}\phi} + \left(\frac{\partial}{\partial x} \pm \mathrm{i}\frac{\partial}{\partial y}\right)\right] \\ a_{\pm 1}^\dagger &= \frac{1}{\sqrt{2}}\left[r\sin\theta\,\mathrm{e}^{\pm\mathrm{i}\phi} - \left(\frac{\partial}{\partial x} \pm \mathrm{i}\frac{\partial}{\partial y}\right)\right] \\ a_0 &= \frac{1}{\sqrt{2}}\left(r\cos\theta + \frac{\partial}{\partial z}\right). \end{aligned} \tag{4}$$

We apply the spherical components $a_{\pm 1}$ and $a_{\pm 1}^\dagger$ to any element of the standard basis,$\Psi_{nlm}(r,\theta,\phi) = Y_{\ell m}(\theta,\phi)R_{n\ell}(r) \equiv Y_{\ell m}(\theta,\phi)f_{n\ell}(r)/r$, and use the recursion relations for the spherical harmonics [16], to obtain

$$\begin{aligned} a_{\pm 1}\Psi_{n\ell m}(r,\theta,\phi) = &-\alpha^{\pm}(\ell,m)Y_{\ell+1,m\pm 1}(\theta,\phi)\frac{1}{r}A_\ell^\dagger f_{n\ell}(r) \\ &- \beta^{\pm}(\ell,m)Y_{\ell-1,m\pm 1}(\theta,\phi)\frac{1}{r}B_\ell^\dagger f_{n\ell}(r) \end{aligned} \tag{5}$$

$$a^\dagger_{\pm1}\Psi_{n\ell m}(r,\theta,\phi) = -\alpha^\pm(\ell,m)Y_{\ell+1,m\pm1}(\theta,\phi)\frac{1}{r}B_{\ell+1}f_{n\ell}(r) - \beta^\pm(\ell,m)Y_{\ell-1,m\pm1}(\theta,\phi)\frac{1}{r}A_{\ell-1}f_{n\ell}(r) \tag{6}$$

$$a_0\Psi_{n\ell m}(r,\theta,\phi) = -\gamma(\ell,m)Y_{\ell+1,m}(\theta,\phi)\frac{1}{r}A^\dagger_\ell f_{n\ell}(r) - \epsilon(\ell,m)Y_{\ell-1,m}(\theta,\phi)\frac{1}{r}B^\dagger_\ell f_{n\ell}(r) \tag{7}$$

where the coefficients $\alpha^\pm(\ell,m)$, $\beta^\pm(\ell,m)$, $\gamma(n,\ell)$ and $\epsilon(n,\ell)$ are

$$\alpha^\pm(\ell,m) = \pm\sqrt{\frac{(\ell\pm m+1)(\ell\pm m+2)}{(2\ell+1)(2\ell+3)}}$$

$$\beta^\pm(\ell,m) = \mp\sqrt{\frac{(\ell\mp m)(\ell\mp m-1)}{(2\ell+1)(2\ell-1)}}$$

$$\gamma(\ell,m) = \sqrt{\frac{(\ell+m+1)(\ell-m+1)}{(2\ell+3)(2\ell+1)}}$$

$$\epsilon(\ell,m) = \sqrt{\frac{(\ell+m)(\ell-m)}{(2\ell+1)(2\ell-1)}}$$

and the operators A_ℓ, $A^\dagger_\ell$, B_ℓ and $B^\dagger_\ell$ are defined as follows:

$$A_\ell = \frac{1}{\sqrt{2}}\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\ell+1}{r} - r\right) \qquad A^\dagger_\ell = \frac{1}{\sqrt{2}}\left(-\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\ell+1}{r} - r\right) \tag{8}$$

$$B_\ell = \frac{1}{\sqrt{2}}\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\ell}{r} - r\right) \qquad B^\dagger_\ell = \frac{1}{\sqrt{2}}\left(-\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\ell}{r} - r\right). \tag{9}$$

Some remarks about the effect of the radial operators A_ℓ, $A^\dagger_\ell$, B_ℓ and $B^\dagger_\ell$ on the reduced wavefunction $f_{n\ell}$ are the following. First, since the operators $a_{\pm1}$ and $a^\dagger_{\pm1}$ are not constants of motion, the radial operators in equations (8) and (9) must change both quantum numbers, n and ℓ, in $f_{n\ell}$. For example, for equation (5) to be self-consistent, the first subindex of the spherical harmonic $Y_{\ell+1,m\pm1}$ and the second one of the resulting radial function $A^\dagger_\ell f_{n\ell}(r)$ must be equal. This implies that the radial operator $A^\dagger_\ell$ must change ℓ to $\ell+1$ in $f_{n\ell}$. These observations are in agreement with those obtained from the purely operational context by Liu *et al* [17], without any reference to creation and annihilation operators. They showed that the radial operators A_ℓ, $A^\dagger_\ell$, B_ℓ and $B^\dagger_\ell$ act on the reduced wavefunction $f_{n\ell}(r)$ as follows:

$$\begin{aligned} A_\ell f_{n\ell}(r) &\propto f_{n-1\ell+1}(r) \qquad & A^\dagger_\ell f_{n\ell}(r) &\propto f_{n-1\ell-1}(r) \\ B_\ell f_{n\ell}(r) &\propto f_{n+1\ell+1}(r) \qquad & B^\dagger_\ell f_{n\ell}(r) &\propto f_{n+1\ell-1}(r). \end{aligned} \tag{10}$$

In the next section we will show that the radial operators A_ℓ, $A^\dagger_\ell$, B_ℓ and $B^\dagger_\ell$ turn out to be the same as those of supersymmetry.

3. Supersymmetry of the 3-DIHO

The one-dimensional radial Schrödinger equation for the 3-DIHO is

$$\left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\ell(\ell+1)}{2r^2} - \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + \frac{r^2}{2}\right]R_{n\ell}(r) = ER_{n\ell}(r) \tag{11}$$

which can be rewritten as

$$H_\ell f_{n\ell}(r) = \left(-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\ell(\ell+1)}{2r^2} + \frac{r^2}{2}\right) f_{n\ell}(r)$$
$$\equiv \left(-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + V_\ell(r)\right) f_{n\ell}(r) = E f_{n\ell}(r). \tag{12}$$

Since supersymmetry in nonrelativistic quantum mechanics is well known [3, 4, 18, 19], we apply its results to our problem. The radial harmonic oscillator is a supersymmetric quantum system with good or broken SUSY, depending on the sign choice of the parameter $\eta \equiv \pm(\ell + 1/2) + 1/2$ [19]. For a fixed ℓ in equation (12), we can obtain the functions

$$u_{0_\pm} = r^\eta \exp\left(-r^2/2\right) \tag{13}$$

which satisfy

$$\frac{1}{2}\frac{u''_{0_\pm}}{u_{0_\pm}} = V_\ell(r) - [\eta + 1/2] \tag{14}$$

and

$$\frac{u'_{0_\pm}}{u_{0_\pm}} = \frac{\eta}{r} - r. \tag{15}$$

For positive values of η, i.e. $\eta = \ell + 1$, SUSY is good, and u_{0+} is the corresponding square-integrable ground-state wavefunction of the problem. For the case of $\eta = -\ell < 0$, SUSY is broken, since the function u_{0_-} cannot be normalized. We note that both functions, $u_{0_\pm}$, lead to the same equations, i.e. equations (14) and (15).

The rate $u'_{0_\pm}/u_{0_\pm}$ defines completely the supersymmetric operators of the problem [3, 4, 18]. From u_{0_+} we obtain supersymmetric operators identical to A_ℓ and $A_\ell^\dagger$. Formally, from u_{0_-}, the resulting SUSY operators are equal to B_ℓ and $B_\ell^\dagger$. This shows that the radial operators of equations (8) and (9), contained in the spherical components of the creation and annihilation operators $a_{\pm 1}$ and $a_{\pm 1}^\dagger$, are the supersymmetric operators for the central potential $V_\ell(r) - [\eta + 1/2]$.

By a straightforward calculation we obtain

$$\begin{aligned} A_\ell A_\ell^\dagger &= H_{\ell+1/2} - (\ell + 3/2) \qquad & A_\ell^\dagger A_\ell &= H_{\ell+3/2} - (\ell + 1/2) \\ B_\ell B_\ell^\dagger &= H_{\ell+1/2} + (\ell - 1/2) \qquad & B_\ell^\dagger B_\ell &= H_{\ell-1/2} + (\ell + 1/2). \end{aligned} \tag{16}$$

From these equations we observe that the operators $A_\ell\,(B_\ell)$ and $A_\ell^\dagger\,\left(B_\ell^\dagger\right)$ are the supersymmetric operators of the Hamiltonian $A_\ell A_\ell^\dagger\,\left(B_\ell B_\ell^\dagger\right)$ and its partner $A_\ell^\dagger A_\ell\,\left(B_\ell^\dagger B_\ell\right)$. This is due to the additional term in the potential of equation (14), which is the reason that the supersymmetric operators do not factorize the Hamiltonian H_ℓ of the 3-DIHO, but the Hamiltonian H_ℓ plus an additional term.

4. Symmetry and factorization of the 3-DIHO

It is well known that the symmetry group of the 3-DIHO is $SU(3)$, whose generators are composed of the components of the orbital angular momentum and those of a rank two symmetric tensor [15, 20, 21]. The constants of motion of the 3-DIHO, written in terms of the spherical creation and annihilation operators (4), are identified with the $SU(3)$-symmetrized operators

$$
\begin{aligned}
&L_0 = a_{-1}^\dagger a_{+1} - a_{+1}^\dagger a_{-1} \qquad L_{\pm 1} = \mp\left(a_{\pm 1}^\dagger a_0 - a_0^\dagger a_{\pm 1}\right) \\
&Q_0 = -\frac{1}{\sqrt{3}}\left(2a_0^\dagger a_0 + a_{-1}^\dagger a_1 - a_1^\dagger a_{-1}\right) \\
&Q_{\pm 1} = \left(a_{\pm 1}^\dagger a_0 + a_0^\dagger a_{\pm 1}\right) \qquad Q_{\pm 2} = -\sqrt{2} a_{\pm 1}^\dagger a_{\pm 1}.
\end{aligned}
\tag{17}
$$

Except for some multiplicative constants, the operators Q_σ, with $\sigma = 0, \pm 1, \pm 2$, are equal to those given in [21]. The components of the rank two tensor can be written as

$$
\begin{aligned}
&Q_0 = 3\left(r^2\cos^2\theta - \frac{\partial^2}{\partial z^2}\right) - H \\
&Q_{\pm 1} = \pm\frac{1}{\sqrt{2}}\left[r^2\cos\theta\sin\theta\,\mathrm{e}^{\pm\mathrm{i}\phi} - \frac{\partial}{\partial z}\left(\frac{\partial}{\partial x} \pm \mathrm{i}\frac{\partial}{\partial y}\right)\right] \\
&Q_{\pm 2} = -\frac{1}{2}\left[r^2\sin^2\theta\,\mathrm{e}^{\pm 2\mathrm{i}\phi} + \left(\frac{\partial}{\partial x} \pm \mathrm{i}\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} \pm \mathrm{i}\frac{\partial}{\partial y}\right)\right]
\end{aligned}
\tag{18}
$$

where H is the Hamiltonian given in equation (2).

A similar procedure to that followed in deriving equations (5)–(7) leads us to

$$
\begin{aligned}
-\frac{1}{3}&Q_0\Psi_{n\ell m}(r,\theta,\phi) \\
&= \Pi(\ell,m)Y_{\ell+2,m}(\theta,\phi)\left(-\frac{2\ell+3}{r}\right)\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\ell+1}{r^2} + \frac{2E}{2\ell+3}\right)f_{n\ell}(r) \\
&\quad + \frac{1}{r}\left(\frac{1}{3} - 2\Upsilon(\ell,m)\right)EY_{\ell,m}(\theta,\phi)f_{n\ell}(r) \\
&\quad + \Xi(\ell,m)Y_{\ell-2,m}(\theta,\phi)\left(\frac{2\ell-1}{r}\right)\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\ell}{r^2} - \frac{2E}{2\ell-1}\right)f_{n\ell}(r)
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
\pm\sqrt{2}&Q_{\pm 1}\Psi_{n\ell m}(r,\theta,\phi) \\
&= \Omega^\pm(\ell,m)Y_{\ell+2,m\pm 1}(\theta,\phi)\left(-\frac{2\ell+3}{r}\right)\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\ell+1}{r^2} + \frac{2E}{2\ell+3}\right)f_{n\ell}(r) \\
&\quad - \frac{2}{r}E\Phi^\pm(\ell,m)Y_{\ell,m\pm 1}(\theta,\phi)f_{n\ell}(r) \\
&\quad + \Sigma^\pm(\ell,m)Y_{\ell-2,m\pm 1}(\theta,\phi)\left(\frac{2\ell-1}{r}\right)\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\ell}{r^2} - \frac{2E}{2\ell-1}\right)f_{n\ell}(r)
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
2&Q_{\pm 2}\Psi_{n\ell m}(r,\theta,\phi) \\
&= \Delta^\pm(\ell,m)Y_{\ell+2,m\pm 2}(\theta,\phi)\left(-\frac{2\ell+3}{r}\right)\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\ell+1}{r^2} + \frac{2E}{2\ell+3}\right)f_{n\ell}(r) \\
&\quad - \frac{2}{r}E\Lambda^\pm(\ell,m)Y_{\ell,m\pm 2}(\theta,\phi)f_{n\ell}(r) \\
&\quad + \Gamma^\pm(\ell,m)Y_{\ell-2,m\pm 2}(\theta,\phi)\left(\frac{2\ell-1}{r}\right)\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\ell}{r^2} - \frac{2E}{2\ell-1}\right)f_{n\ell}(r)
\end{aligned}
\tag{21}
$$

where the coefficients $\Pi(\ell,m)$, $\Upsilon(\ell,m)$, $\Xi(\ell,m)$, $\Omega^{\pm}(\ell,m)$, $\Phi^{\pm}(\ell,m)$, $\Sigma^{\pm}(\ell,m)$, $\Delta^{\pm}(\ell,m)$, $\Lambda^{\pm}(\ell,m)$ and $\Gamma^{\pm}(\ell,m)$ are

$$\Pi(\ell,m) = \sqrt{\frac{(\ell+m+1)(\ell-m+1)(\ell+m+2)(\ell-m+2)}{(2\ell+5)(2\ell+3)^2(2\ell+1)}}$$

$$\Upsilon(\ell,m) = \frac{2\ell^2+2\ell-2m^2-1}{(2\ell+3)(2\ell-1)}$$

$$\Xi(\ell,m) = \sqrt{\frac{(\ell+m)(\ell-m)(\ell+m-1)(\ell-m-1)}{(2\ell+1)(2\ell-1)^2(2\ell-3)}}$$

$$\Omega^{\pm}(\ell,m) = \pm\sqrt{\frac{(\ell\pm m+3)(\ell\pm m+2)(\ell\pm m+1)(\ell\mp m+1)}{(2\ell+5)(2\ell+3)^2(2\ell+1)}}$$

$$\Phi^{\pm}(\ell,m) = \pm\frac{2m\pm 1}{(2\ell+3)(2\ell-1)}\sqrt{(\ell\mp m)(\ell\pm m+1)}$$

$$\Sigma^{\pm}(\ell,m) = \mp\sqrt{\frac{(\ell+m)(\ell-m)(\ell\mp m-1)(\ell\mp m-2)}{(2\ell+1)(2\ell-1)^2(2\ell-3)}}$$

$$\Delta^{\pm}(\ell,m) = \sqrt{\frac{(\ell\pm m+4)(\ell\pm m+3)(\ell\pm m+2)(\ell\pm m+1)}{(2\ell+5)(2\ell+3)^2(2\ell+1)}}$$

$$\Lambda^{\pm}(\ell,m) = \frac{-2}{(2\ell+3)(2\ell-1)}\sqrt{(\ell\pm m+2)(\ell\pm m+1)(\ell\mp m)(\ell\mp m-1)}$$

$$\Gamma^{\pm}(\ell,m) = \sqrt{\frac{(\ell\mp m)(\ell\mp m-1)(\ell\mp m-2)(\ell\mp m-3)}{(2\ell+1)(2\ell-1)^2(2\ell-3)}}.$$

We note that the operators of equations (18) contain second-order derivatives. So, the radial operators in equations (19)–(21) were obtained by using the radial Schrödinger equation (12) to transform second-order radial derivatives into first-order ones. This is the reason why the energy E is now involved in these expressions. This procedure has been used in a previous work by studying the two-dimensional harmonic oscillator [8].

Since the operators Q_0, $Q_{\pm1}$ and $Q_{\pm2}$ are tensor operators [15], they must transform the state $\Psi_{n\ell m}(r,\theta,\phi)$ into a linear combination of states belonging to the same energy level. Because of this, the radial operators in equations (19)–(21) must transform $f_{n\ell}(r)$ into $f_{n\ell\pm2}(r)$, or into a factor times $f_{n\ell}(r)$, depending on whether the ℓ-subindex value of the companion spherical harmonic is $\ell\pm2$ or ℓ, respectively. So, the operators that increase or decrease the angular momentum by 2, but keep the energy E unchanged, are

$$\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}+\frac{\ell}{r^2}-\frac{2E}{2\ell-1}\right)f_{n\ell}(r)\propto f_{n,\ell-2} \qquad \left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}-\frac{\ell+1}{r^2}+\frac{2E}{2\ell+3}\right)f_{n\ell}(r)\propto f_{n,\ell+2} \tag{22}$$

respectively. These radial operators were obtained by Liu *et al* [17] from a purely operational context by factorizing the radial equation (12) without relating them to the constants of motion. Since we can show that equation (12) cannot be factorized by means of the IH factorization method [2], the factorization technique used in [17] is different to that of IH. However, we can show that equation (11) and the definitions $G(r) = r^{3/2}R(r)$ and $x = r^2$, lead us to

$$\left(\frac{d^2}{dx^2} - \frac{(\ell/2 + 3/4)(\ell/2 - 1/4)}{x^2} + \frac{2E}{x} - \frac{1}{4}\right) G(x) = 0. \tag{23}$$

This equation is classified by the IH factorization method as one of class I type F [2], and its factorization operators can immediately be found. We can show that these IH factorization operators are equal to those of equation (22).

5. Concluding remarks

The constants of motion of the two-dimensional problems of the hydrogen atom and the harmonic oscillator have been obtained from SUSY operators [13, 14]. This is because in the two-dimensional cases, due to the simplicity of the angular dependence of the wavefunction, it is easy to replace the angular momentum quantum number m by a differential operator. However, in our problem, where the angular dependence of the wavefunction is given by the spherical harmonics, the procedure followed in [13, 14] is not so easy to apply to obtain the operators Q_0, $Q_{\pm 1}$ and $Q_{\pm 2}$ from supersymmetric operators.

For the 3-DIHO we have shown that the radial part of the spherical creation and annihilation operators and their adjoints, are equal to its supersymmetric operators. Also, we showed that the constants of motion of the problem lead us to second-order radial operators which are equal to those obtained by the IH factorization method, and change the orbital angular momentum quantum number by two units. Moreover, the subtle relationship between the IH radial operators and the SUSY operators is obtained from the bilinear combinations of equations (17). At this stage, it is interesting to note that the spherical creation and annihilation operators (4) as well as the constants of motion (17) contain the good and broken SUSY operators, given by equations (8) and (9), respectively.

Finally, in this work we have treated a problem that because of its complexity was not explicitly studied in [11], where the relation between closeness of the orbits in classical mechanics and the radial factorization in quantum mechanics was analysed.

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